Algebra II — Lesson 0812 - Solving Linear Equations
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1. Solutions live in a specified set of numbers

An equation is a statement. It is either true or false. If the equation contains a variable or variables, the truth or falsity of the equation may depend on the value(s) of the variable(s).

Consider an equation with a single variable x. To *solve* this equation in a given set of numbers means to find all numbers in that set which, when plugged in for x, make the equation true. Such an x is said to *satisfy* the equation. Whether or not an equation has a solution may depend on what type of solution we are looking for. We may be looking for a positive integer as a solution, or we may be happy to find an integer, rational, or real solution.

We give examples. The equation $x + 3 = 5$ has a solution (which is $x = 2$) in $\mathbb{N}$. The equation $x + 5 = 3$ has no solution in $\mathbb{N}$, but does have a solution (which is $x = -2$) in $\mathbb{Z}$. The equation $2x = 1$ has no solution in $\mathbb{Z}$, but does have a solution (which is $x = \frac{1}{2}$) in $\mathbb{Q}$. The equation $x^2 - 2 = 0$ has no solution in $\mathbb{Q}$, but has two solution in $\mathbb{R}$, those being $x = \sqrt{2}$ or $x = -\sqrt{2}$. Does the equation $x^2 + 2 = 0$ have a solution?

2. Linear Equations

A *linear equation* is an equation that can be put in the form

$$ax + b = 0,$$

where a and b are numbers, and x is a variable.

We solve some of these.

Example 1. Solve the equation $3x + 8 = 17$.

Solution. First subtract 8 from both sides to get $3x = 9$. Then, divide by 3 to get $x = 3$. Write your work by lining up the equal signs:

$$\begin{aligned} 3x + 8 &= 17 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

When you subtract 8 from both sides, don't write -8 and -8 under both sides of the equation. Just subtract 17-8=9 in your head. $\square$

Example 2. Find all $x \in \mathbb{Z}$ such that $5x - 3 = 9x + 10$.

Solution. This is still a linear equation, because it can be put in the form $ax + b = 0$ (even though it isn't in that form yet). The variable is x; the numbers 5, -3, 9, and 10 are called *coefficients*.

$$\begin{aligned} 5x - 3 &= 9x + 10 && \\ 9x + 10 &= 5x - 3 && \text{switch sides to get the bigger} \\ 4x + 10 &= -3 && \text{subtract } 5x \text{ from both sides} \\ 4x &= -7 && \text{subtract 10 from both sides} \\ x &= -\frac{7}{4} && \text{divide both sides by 4} \end{aligned}$$

We found the only solution to be $-\frac{7}{4}$, which is rational but is not an integer; thus, there are no solutions in $\mathbb{Z}$ (but there is one in $\mathbb{Q}$). $\square$

A linear equation with real coefficients always has exactly one solution in $\mathbb{R}$. We solve a general linear equation.

Example 3. Solve $ax + b = 0$.

Solution. Subtract b from both sides to get $ax = -b$. Divide both sides by a to get $x = -\frac{b}{a}$. You may write this as a sequence of equations if you wish.

$$\begin{aligned} ax + b &= 0 \\ ax &= -b && \text{subtract } b \text{ from both sides} \\ x &= -\frac{b}{a} && \text{divide both sides by } a \end{aligned}$$

$\square$

If the coefficients of a linear equation are rational, then the solution is also rational.

If the coefficients of a linear equation are integers, the solution may be a fraction.

Example 4. For each linear equation, solve it, and determine the smallest set among $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ in which the solution resides.

(a) $3x - 5 = 1$. Then $3x = 6$, so $x = 2$. Then $x \in \mathbb{N}$.
(b) $x + 8 = 3$. Then $x = -11$, so $x \in \mathbb{Z}$, but $x \notin \mathbb{N}$.
(c) $7x + 2 = x + 5$. Then $6x + 2 = 5$, so $6x = 3$, whence $x = \frac{3}{6} = \frac{1}{2}$. Thus $x \in \mathbb{Q}$, but $x \notin \mathbb{Z}$.

3. Exercises

Problem 1. For each linear equation, solve it, and determine the smallest set among $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ in which the solution resides.

(a) $4x + 8 = 0$
(b) $3x + 3 = 2x - 6$
(c) $5x + 101 = 7x + 57$
(d) $\pi x = \pi^2$